Suggested Solution to MATH206B Home Test 3

(CCWLeung): 14 May 2020

Q1a:

Fix $x, h \in \mathbb{R}$ with $h \neq 0$.

Then by the ~~Taga~~ Taylor expansion of f, we have:

$$\begin{cases} f(x+h) = f(x) + f'(x)h + \dfrac{f''(c_1)}{2!}h^2 & \cdots (1) \\ f(x-h) = f(x) - f'(x)h + \dfrac{f''(c_2)}{2!}h^2 & \cdots (2) \end{cases}$$

for some c_1, c_2 between x and $x+h$

(resp. x and $x-h$)

By considering (1) + (2), we see that

$$|f(x+h) + f(x-h) - 2f(x)|$$

$$\leq \frac{|f''(c_1)|}{2} h^2 + \frac{1}{2}|f''(c_2)|h^2 \leq C h^2$$

$\square$

Q1b: Note that since f is cont,

the function $F(x) \triangleq \int_a^x f(t)\,dt$ is differentiable

with $F'(x) = f(x)$.

Then $\displaystyle\int_a^b \frac{f(x+h) - f(x)}{h}\,dx = \int_a^b \frac{F'(x+h) - F'(x)}{h}\,dx$

$$= \frac{F(b+h) - F(a+h) - (F(b) - F(a))}{h}$$

$\boxed{\text{(Q1b)}}$:

$$\therefore \lim_{h\to 0} \int_a^b \frac{f(x+h)-f(x)}{h}\,dx$$

$$= \lim_{h\to 0}\left[\frac{F(b+h)-F(b)}{h} - \frac{F(a+h)-F(a)}{h}\right]$$

$$= F'(b) - F'(a) = f(b) - f(a)$$

$\square$

$\boxed{\text{Q1C}}$:

For each $x>0$, put $M_x \equiv \sup_{t\in[0,x]} |f(t)|$

Note that since f is cont, $M_x < \infty$, $\forall x>0$

Also, we see that

$$f\equiv 0 \text{ on } [0,x] \text{ iff } M_x = 0$$

By using mean value th, ~~$\forall x>0$~~ for $0<t\leq x$, $\exists\ c_t \in (0,t)$

s.t. $|f(t)| = |f(t)-f(0)| = |f'(c_t)(t-0)| \leq |f(c_t)|\,t \leq M_x t$

$\Rightarrow |f(t)| \leq M_x t \leq M_x x$, $\forall t\in(0,x]$

$\Rightarrow M_x \leq M_x x$, $\forall x>0$

$\therefore$ if $M_x > 0 \Rightarrow x\geq 1$

$\therefore$ if $0\leq x<1 \Rightarrow f(x)\equiv 0$

By the continuity of $f \Rightarrow f|_{[0,1]} \equiv 0$

Q 1C

Let $g_1(x) \equiv f(x+1)$

$\Rightarrow \quad g_1(0) = 0$ and $|g_1'(x)| \le |g_1(x)|, \forall x > 0$

$\therefore \quad g_1|_{[0,1]} \equiv 0 \Rightarrow f|_{[1,2]} \equiv 0$

To repeat the same step, if we

consider
$$g_n(x) \equiv f(x+n)$$

then $\quad f|_{[0,n]} \equiv 0, \quad \forall\ n = 1, 2 \ldots$

$\Rightarrow \quad f \equiv 0 \quad \forall x \ge 0 \quad \square$

Q2(i)

It suffices to show that if $\lim b_n = 0 \Rightarrow$

$$\lim_{N} \frac{1}{N} \sum_{n=1}^{N} b_n = 0$$

Let $\varepsilon > 0$. Since $\lim_{n} b_n = 0$, $\exists C > 0$, $\exists N_1$

s.t. $\begin{cases} |b_n| < \varepsilon, & \forall n \geq N_1 \quad \text{and} \\ |b_n| < C, & \forall n \end{cases}$

Now choose N_2 s.t.

$$\frac{N_1 C}{N} < \varepsilon, \quad \forall N > N_2$$

Then for any $N > N_2$, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} b_n \right| \leq \frac{1}{N} \sum_{n=1}^{N_1} |b_n| + \frac{1}{N} \sum_{N_1 < n \leq N} |b_n|$$

$$< \frac{N_1 C}{N} + \frac{(N - N_1)}{N} \varepsilon < 2\varepsilon$$

$\square$

$\boxed{Q\,2\,(ii)}$ Let $\varepsilon > 0$. By part (i), $\exists N_1$

s.t. $\frac{1}{N} \sum_{k=1}^{N} |k a_k| < \varepsilon, \quad \forall N > N_1$

Note: $N(x) \uparrow \infty$ iff $x \to 1-$

$\therefore \exists \delta > 0$ s.t.

$N(x) > N_1, \quad \forall 1-\delta < x < 1,$

Then we have

$$\left| \sum_{k=1}^{N(x)} a_k (1-x^k) \right| = \left| \sum_{k=1}^{N} a_k (1-x)(1+x+\dots+x^{k-1}) \right|$$

$$\leq (1-x) \sum_{k=1}^{N(x)} |a_k| k$$

$$= (1-x) N(x) \cdot \frac{1}{N(x)} \sum_{k=1}^{N(x)} k |a_k|$$

$$\leq \varepsilon \quad (\because (1-x) N(x) \leq 1)$$

whenever $1-\delta < x < 1$.

$$\therefore \lim_{x \to 1-} \sum_{k=1}^{N(x)} a_k (1-x^k) = 0 \quad \text{-- } (*)$$

$\boxed{Q2(ii):}$ $\vdash$: $\lim_{x \to 1-} \sum_{k=1}^{N(x)} a_k = R$

(pf $\vdash$): Let $\varepsilon > 0$. Since $\lim_{k} k a_k = 0$, $\exists N$ s.t

$$|a_k| < \frac{\varepsilon}{N}, \quad \forall k > N$$

Also, since $\lim_{x \to 1-} \sum_{k=1}^{N(x)} a_k (1 - x^k) = 0$ and

$$R \equiv \lim_{x \to 1-} \sum_{k=1}^{\infty} a_k x^k = R,$$

$\therefore$ $\exists\ \delta > 0$ s.t

$$\begin{cases} N(x) > N \\ \left|R - \sum_{k=1}^{\infty} a_k x^k\right| < \varepsilon \\ \left| \sum_{k=1}^{N(x)} a_k (1 - x^k) \right| < \varepsilon \end{cases}$$

as $1 - \delta < x < 1$.

$\boxed{\text{Q2(ii)}}$ Then for $1-\delta < x < 1$, we have

$$|R - \sum_{k=1}^{N(x)} a_k| \le |R - \sum_{k=1}^{\infty} a_k x^k| + |\sum_{k=1}^{\infty} a_k x^k - \sum_{k=1}^{N(x)} a_k|$$

$$< \varepsilon + |\sum_{k=1}^{\infty} a_k (1-x^k)|$$

$$< \varepsilon + |\sum_{k=1}^{N(x)} a_k(1-x^k)| + |\sum_{k > N(x)}^{\infty} a_k x^k|$$

$$< \varepsilon + \varepsilon + \frac{\varepsilon}{N(x)} \sum_{k > N(x)} x^k \quad \left(\because |a_k| < \frac{\varepsilon}{k} < \frac{\varepsilon}{N(x)} \ \forall k \ge N(x) > N \right)$$

$$< 2\varepsilon + \frac{\varepsilon}{N(x)} \cdot \frac{1}{(1-x)}$$

$$< 2\varepsilon + \left(\frac{1+N(x)}{N(x)}\right) \varepsilon$$

$$\left(\because N(x) \le \frac{1}{1-x} < 1 + N(x) \right)$$

$$< 2\varepsilon + \varepsilon = 3\varepsilon$$

$\square$

Q2(iii): $\vdash \sum_{k=1}^{\infty} a_k = R$

(pf $\vdash$): By (ii), $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|R - \sum_{k=1}^{N(x)} a_k| < \varepsilon. \quad \text{as} \quad 1 - \delta < x < 1$$

Consider $x_n \equiv 1 - \frac{1}{n}$, $n = 1, 2 \ldots$

Then $N(x_n) = n$, $\forall n = 1, 2 \ldots$

$\therefore$ Choose N s.t.

$$1 - \delta < x_n < 1, \quad \forall n \geq N.$$

Then we have

$$|R - \sum_{k=1}^{n} a_k| = |R - \sum_{k=1}^{N(x_n)} a_k| < \varepsilon$$

as $n \geq N$.

$\square$

$\boxed{3a}$

$\overline{\text{i}}$: $f \in R[a,b]$.

(pf $\overline{\text{i}}$): Let $\varepsilon > 0$. Choose $a < c < d < b$ st.

$b - d < \varepsilon$ and $c - a < \varepsilon$

$\because$ $|f_n(x)| \le M, \ \forall n, \ \forall x \in [a,b]$

$\therefore$ We have $|f(x)| \le M, \ \forall x \in [a,b]$

On the other hand, by the assumption, $f_n \to f$ uniformly on $[c,d]$. then

$f \in R[c,d]$ and $\lim_n \int_c^d f_n = \int_c^d f$

$\therefore$ $\exists$ partation P_1 on $[c,d]$ st

$$\sum \omega_i (f|_{[c,d]}, P_1)\, \Delta x_i < \varepsilon$$

Now if we let $P \equiv P_1 \cup \{a,b\}$,

then P is a partation $[a,b]$ and

$$\sum_i \omega_i(f,P)\,\Delta x_i = \omega_1(f,P)(c-a) + \sum_{x_i \neq a,b} \omega_i(f,P)\,\Delta x_i + \omega_n(f,P)(b-d)$$

$$< 2M\varepsilon + \varepsilon + 2M\varepsilon$$

$\therefore$ $f \in R[a,b]$ $\square$

9

$\boxed{3a}$ I_2: $\lim_n \int_a^b f_n = \int_a^b f$

Pf I_2: Let $\varepsilon > 0$. Let c, d be as in (I_1)

$\therefore$ $f_n \to f$ uniformly on $[c, d]$,

$\therefore$ $\exists$ N s.t. $\left| \int_c^d f_n - \int_c^d f \right| < \varepsilon$, $\forall n \geq N$

Then for any $n \geq N$, we have

$$\left| \int_a^b f_n - \int_a^b f \right| \leq \int_a^c |f_n - f| + \left| \int_c^d f_n - \int_c^d f \right| + \int_d^b |f_n - f|$$

$$\leq 2M\varepsilon + \varepsilon + 2M\varepsilon.$$

$\square$

10

3b(i)

$$\because \sum_{k=1}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

$$\therefore \underbrace{\int_0^{x}\left(\sum_{k=1}^{n}\cos kt\right)dt}_{= f_n(x)} = \int_0^{x}\left(\frac{\sin(n+\frac{1}{2})t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\right)dt \quad \forall x \in [0, \pi].$$

We are going to estimate

$$\underbrace{\quad}_{I_n(x)} \overset{\text{def}}{=} \int_0^{x}\left(\frac{\sin(n+\frac{1}{2})t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\right)dt \quad \text{as } n \to \infty$$

Note:

$$I_n(x) = \int_0^{x}\frac{\sin(n+\frac{1}{2})t}{t}\,dt$$

$$+ \int_0^{x}\sin(n+\tfrac{1}{2}t)\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)dt$$

$$- \int_0^{x}\frac{\sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\,dt$$

$$(\star) \;\; \overset{}{=} \int_0^{(n+\frac{1}{2})x}\frac{\sin y}{y}\,dy + \int_0^{x}\sin(n+\tfrac{1}{2})t\left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right]dt$$

(Let $y = (n+\frac{1}{2})t$) ~~(this is an improper integral)~~

$$- \frac{1}{2}x, \quad \forall n, \; \forall x \in [0, \pi]$$

11

$\boxed{3b\ (i)}$

Note: $\lim_{n\to\infty} \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y}\,dy = \int_0^{\infty} \frac{\sin y}{y}\,dy = \frac{\pi}{2}$,

$\forall\ 0 < x \leq \pi$

On the other hand, we have

$$\int_0^x \sin(n+\tfrac{1}{2})t \left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right] dt$$

$$= -\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) \frac{\cos(n+\frac{1}{2})t}{n+\frac{1}{2}}\Bigg|_0^x$$

$$+ \left(\frac{1}{n+\frac{1}{2}}\right) \int_0^x \cos(n+\tfrac{1}{2})t \; d\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)$$

$\longrightarrow 0$ as $n \to \infty$

$\therefore$ by ($\star$) in (ii) above, we have

$$\lim_n f_n(x) = \lim_n I_n(x)$$

$$= \begin{cases} \frac{1}{2}\pi - \frac{x}{2} & 0 < x \leq \pi \\ 0 & x = 0 \end{cases}$$

$\square$

$\boxed{3\ b\ (ii)}$

Ans : No.

By (i)

$$f(x) = \lim_n f_n(x) = \begin{cases} \frac{\pi}{2} - \frac{x}{2} & 0 < x \leq \pi \\ 0 & x = 0 \end{cases}$$

Note that each f_n is cont on $[0, \pi]$

but f is discont at $x = 0$

$\therefore$ (f_n) does not conv uniformly on $[0, \pi]$.

$\boxed{3b\,(iii)}$

$(\dagger)$: $\exists M > 0$ st $|f_n(x)| \le M$, $\forall n$ and $\forall x \in [0, \pi]$

~~Note that~~

(pf $\dagger$): Recall Eq($\star$) in (i), we have

$$f_n(x) = \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y}\,dy + \int_0^x \sin(n+\tfrac{1}{2})t \left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right] dt$$

$$- \frac{1}{2}x$$

N.B.: $0 \le \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y}\,dy \le \int_0^{\infty} \frac{\sin y}{y}\,dy = \frac{\pi}{2}$

$\forall n, \forall x \in [0, \pi]$

and

$$0 \le \int_0^{x} \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \cdot \sin(n+\tfrac{1}{2})t \, dt$$

$(\because \sin z \le z \ \forall z)$ $\le \int_0^{\pi} \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) dt < \infty$

~~(as this improper integral is convergent)~~

$\forall x \in [0, \pi]$, $\forall n$

$\therefore$ $(\dagger)$ follows $\square$

$\boxed{3b(iii)}$

$\frac{1}{2}$: (f_n) cnv uniformly on $[c,\pi]$, $\forall 0<c<\pi$

(pf $\frac{1}{2}$):

$\because f_n(x) = \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y}\,dy + \int_0^x \sin(n+\frac{1}{2})t\left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right]dt$

$-\frac{1}{2}x$.

$\therefore$ We need to show:

$h_n(x) \equiv \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y}\,dy$ and

$g_n(x) \equiv \int_0^x \sin(n+\frac{1}{2})t\left[\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right]dt$

cnv uniformly on $[c,\pi]$, (fix $0<c<\pi$)

$\frac{1}{2}'$: $(h_n(x))$ cnv uniformly on $[c,\pi]$

(pf $\frac{1}{2}'$): N.B: $\lim_n h_n(x) = \int_0^\infty \frac{\sin y}{y}\,dy = \frac{\pi}{2}$.

$\forall x \in [c,\pi]$

$\therefore \forall \varepsilon>0, \exists M>0$ s.t

$\left|\frac{\pi}{2} - \int_0^T \frac{\sin y}{y}\,dy\right| < \varepsilon$, $\forall T \geq M$

$\therefore$ if we choose N s.t $\forall n \geq N$ we have

$(n+\frac{1}{2})c > M$, ~~[illegible]~~

$\Rightarrow (n+\frac{1}{2})x \geq (n+\frac{1}{2})c \geq (N+\frac{1}{2})c > M$

$\forall n \geq N$, $\forall x \in [c,\pi]$.

3b (iii)

$$\Rightarrow \left| \frac{\pi}{2} - \int_0^{(n+\frac{1}{2})x} \frac{\sin y}{y} dy \right| < \varepsilon$$

$\therefore$ $h_n \to \frac{\pi}{2}$ conv uniformly on $[c, \pi]$

Cl 2'': $g_n \to 0$ uniformly on $[c, \pi]$

(pf cl 2''): Using integration by part, we have

$g_n(x) =$

$$-\left(\frac{1}{2\sin\frac{1}{2}x} - \frac{1}{x}\right) \frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}}$$

$$+ \frac{1}{n+\frac{1}{2}} \int_0^x d\left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right) \cos(n+\tfrac{1}{2})t \, dt$$

Note that the fuction

$t \in [0, x] \mapsto d\left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right)'$ is cont

$\therefore$ $g_n(x) \to 0$ uniformly on

$[c, \pi]$

$\square$ II

3b (iii)

$\therefore$ We can apply part (a), we have

$$\lim_{n} \int_0^{\pi} f_n(x)\,dx = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} \left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= \frac{\pi^2}{4}$$

$\square$